18

COLLIDING ELECTROMAGNETIC WAVES: DIAGONAL SOLUTIONS

This chapter is a continuation of the review of all presently known exact solutions which describe the collision of plane electromagnetic waves, or a combination of both gravitational and electromagnetic waves. Attention is concentrated here only on diagonal solutions. These solutions may be considered as a generalization of the solutions representing the collision of gravitational waves with colinear aligned polarization that have been described in Chapter 10. It may be mentioned that the Bell–Szekeres solution that has previously been described in Chapter 15 also belongs to the class that is now being surveyed.

18.1 The generation technique of Panov

Panov (1978, 1979*a*) has obtained a class of exact solutions for colliding gravitational-electromagnetic waves using a generating technique due to Enss (1967). His method starts with a vacuum solution with aligned polarization, and can conveniently be described as follows.

Taking the metric in the form of the Szekeres line-element (6.20), if a solution of (6.22) describing a collision of plane colinear gravitational waves is given by U_o, V_o, and M_o with $W_\mathrm{o}=0$, then a new solution with non-zero electromagnetic components is given by

$$\begin{aligned} U &= U_\mathrm{o}, \qquad\qquad W = W_\mathrm{o} = 0 \\ e^V &= e^{V_\mathrm{o}}\left(\cos^2\alpha+\sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)^2 \\ e^{-M} &= e^{-M_\mathrm{o}}\left(\cos^2\alpha+\sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)^2 \end{aligned} \tag{18.1}$$

where α is an arbitrary constant. The Maxwell components are given by

$$\begin{aligned} \Phi_0^\circ &= -i\frac{\sin\alpha\cos\alpha e^{-(U_\mathrm{o}+V_\mathrm{o})/2}}{\left(\cos^2\alpha+\sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)}(U_{\mathrm{o}v}+V_{\mathrm{o}v}) \\ \Phi_2^\circ &= -i\frac{\sin\alpha\cos\alpha e^{-(U_\mathrm{o}+V_\mathrm{o})/2}}{\left(\cos^2\alpha+\sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)}(U_{\mathrm{o}u}+V_{\mathrm{o}u}). \end{aligned} \tag{18.2}$$

It may immediately be seen from these equations that the boundary conditions for colliding plane waves are automatically satisfied for the new solution if they were satisfied for the original vacuum solution. The new solution therefore describes the collision of combined gravitational and electromagnetic waves with aligned polarization. The electromagnetic components vanish and the solution reverts to the original when $\alpha = 0$.

In terms of the notation developed in Chapter 16, the initial solution may be described either by the function $\chi_{\rm o}$ with $\omega_{\rm o} = 0$, or by $\Psi_{\rm o}$ with $\Phi_{\rm o} = 0$. The new solution is then given by

$$\chi = \chi_{\rm o} \left(\cos^2 \alpha + \sin^2 \alpha \, e^{-U_{\rm o}} \chi_{\rm o}\right)^{-2}, \qquad \omega = 0, \tag{18.3}$$

and the potential H can be obtained by integrating the equations

$$\begin{aligned} H_v &= i \sin \alpha \cos \alpha \, (U_{{\rm o}v} + V_{{\rm o}v}) \, e^{-U_{\rm o}} \\ H_u &= -i \sin \alpha \cos \alpha \, (U_{{\rm o}u} + V_{{\rm o}u}) \, e^{-U_{\rm o}}. \end{aligned} \tag{18.4}$$

Since H is purely imaginary, the new solution may also be described in terms of the potentials

$$\Psi = \Psi_{\rm o} \left(\cos^2 \alpha + \sin^2 \alpha \, e^{-2U_{\rm o}} \Psi_{\rm o}^{-1}\right)^2, \qquad \Phi = 0. \tag{18.5}$$

Hence the potential Z may be determined using (16.19).

In his original paper, Panov (1978) applied this transformation to the Khan–Penrose solution. In his second paper (1979*a*), he applied it to the general solution given by Szekeres (1972) as described in Section 14.2. In these solutions there is always a space-like curvature singularity on the surface given by $e^{-U} = f + g = 0$ on which the opposing waves mutually focus each other.

The technique described in this section will be referred to again in the following section. It will be shown there how other solutions that do not contain strong curvature singularities, such as the Bell–Szekeres solution, may be included in the notation of this section.

18.2 An alternative approach

The class of solutions in which the metric can be diagonalized is characterized by the fact that the Ernst potentials Z and E are real and that $\omega = 0$. In terms of the functions of the Szekeres line element (6.20), the equivalent condition is that $W = 0$, and hence from (6.22*e*), that $\Phi_0\bar{\Phi}_2 = \Phi_2\bar{\Phi}_0$. In the general case, the components Φ_0, Φ_2 and H or η must remain complex.

In this case the main field equations (16.6) putting $\chi = e^{-V}$, or $(6.22\,d, e)$ expressed in terms of the potential H, together with Maxwell's equations (16.7), can be written in the form

$$\begin{aligned} 2V_{uv} - U_uV_v - U_vV_u + 2e^{U-V}(H_u\bar{H}_v + \bar{H}_uH_v) &= 0 \\ H_u\bar{H}_v - \bar{H}_uH_v &= 0 \\ 2H_{uv} - V_uH_v - V_vH_u &= 0 \end{aligned} \tag{18.6}$$

In the particular case when the polarization vectors of the approaching gravitational waves and electromagnetic waves are all aligned, it is possible to use a duality transformation to ensure that Φ_0, Φ_2 and thus H or η are all real.

In order to generate exact solutions of this type, it is appropriate to consider the technique described at the end of Section 17.4. Accordingly, we consider an initial solution of the vacuum field equations describing the collision of plane gravitational waves. This may be described in terms of the initial Ernst potential E_o which is now real. The potentials for a new diagonal solution describing the collision of plane electromagnetic waves, usually coupled to gravitational waves, are then given by (17.32) or (17.34), where a, b, c and d are all real and satisfy (17.33). Such solutions clearly revert to the original vacuum solution when $a = 1$, $b = 0$, $c = 0$ and $d = 0$.

It is of interest initially to consider the restricted class of solutions that can be obtained from (17.32) when $b = 0$ and $d = 0$. In this case $c = \sqrt{1 - a^2}$ and the transformation reduces to (17.24). Then, reverting to the alternative parameter α given by (17.25), the new solution is given partly as in (17.27) by

$$\begin{gathered} Z = \frac{\sin^2\alpha + \cos^2\alpha\, Z_\mathrm{o}}{\cos^2\alpha + \sin^2\alpha\, Z_\mathrm{o}} \qquad H = \frac{\sin\alpha\cos\alpha\,(Z_\mathrm{o} - 1)}{\cos^2\alpha + \sin^2\alpha\, Z_\mathrm{o}} \\ \Psi = Z - H^2 = Z_\mathrm{o}(\cos^2\alpha + \sin^2\alpha\, Z_\mathrm{o})^{-2} \end{gathered} \tag{18.7}$$

The metric functions V or χ are thus given by

$$e^{-V} = \chi = e^{-V_\mathrm{o}}\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\mathrm{o}-V_\mathrm{o})}\right)^2. \tag{18.8}$$

This expression can be seen to be indentical to that of (18.1) apart from an insignificant change in the sign of V. It can therefore be seen that the technique described by (17.24) in the diagonal case is identical to that of Panov (1978, 1979*a*) that has already been described in the previous section. The complete solution in this case is given exactly as in (18.1) and (18.2) but with a change of sign for V and V_o.

It has been noted in the previous section that solutions obtained using the general technique (17.32) automatically satisfy the boundary conditions for colliding electromagnetic and gravitational waves if those conditions are satisfied for the initial vacuum solution. In the particular case of Panov, as described in Section 18.1, this has already been noted.

It is important to notice that we are here using the Ernst potential rather than the metric functions. Thus using (11.3), (11.11) and (12.36), the functions associated with the initial solution are related by

$$\frac{1+E_\mathrm{o}}{1-E_\mathrm{o}}=Z_\mathrm{o}=\Psi_\mathrm{o}=\frac{\sqrt{1-t^2}\sqrt{1-z^2}}{\chi_\mathrm{o}}=e^{-U_\mathrm{o}}e^{V_\mathrm{o}}. \tag{18.9}$$

Using these identities, the main field equations may be seen to reduce to the single real linear equation

$$\big((1-t^2)V_{\mathrm{o},t}\big)_{,t}-\big((1-z^2)V_{\mathrm{o},z}\big)_{,z}=0 \tag{18.10}$$

which may be seen to be identical to (10.13), which is the main equation for vacuum solutions with aligned polarization. A general solution of this equation will contain the terms (10.16), (10.20) and (10.22), and can be written in the form

$$\begin{aligned}V_\mathrm{o}=&\sum_n\big[a_nP_n(t)P_n(z)+q_nQ_n(t)P_n(z)+p_nP_n(t)Q_n(z)\\&+b_nQ_n(t)Q_n(z)\big]-\tfrac{1}{2}c\log(1-t^2)-\tfrac{1}{2}c\log(1-z^2)\\&+\sum_i d_i\cosh^{-1}\left(\frac{c_i-tz}{\sqrt{1-t^2}\sqrt{1-z^2}}\right)\end{aligned} \tag{18.11}$$

where $P_n(x)$ and $Q_n(x)$ are Legendre functions of the first and second kinds respectively, and a_n, p_n, q_n, b_n c_i and d_i are sets of arbitrary constants.

In his first paper, Panov (1978) derived a new solution using the above technique with $b=0$ and $d=0$ and with the initial solution taken to be the Khan–Penrose solution given as in (10.18) by

$$V_\mathrm{o}=-2Q_0(t)P_0(z). \tag{18.12}$$

In his second paper (1979*a*), he applied the same technique to the general solution given by Szekeres (1972).

It may also be observed using (16.31) that the Bell–Szekeres solution can be obtained with this method by setting $a=0$, $b=0$, $c=1$ and $d=0$ and using the initial solution of (18.8) given by

$$\begin{aligned}V_\mathrm{o}&=-2P_0(t)Q_0(z)-\tfrac{1}{2}\log(1-t^2)-\tfrac{1}{2}\log(1-z^2)\\&=-2P_0(t)Q_0(z)+U_\mathrm{o}.\end{aligned} \tag{18.13}$$

However, as noted in (16.32,33), the same solution is also obtained if the term $2P_0(t)Q_0(z)$ is replaced by $2Q_0(t)P_0(z)$. It can then be seen from (10.35) that the initial solution, in this case, is the degenerate Ferrari–Ibañez solution described in Section 10.5.

In their substantial paper that has already been frequently quoted, Chandrasekhar and Xanthopoulos (1987*a*) have also described a general class of diagonal solutions. This has also been obtained using the above approach but with the constraint that $a=0$, $b=0$, $c=1$ and $d=0$ so that $Z=1$. In this case, it is convenient to introduce a new function F such that

$$F = \tfrac{1}{2}(V_\mathrm{o} - U_\mathrm{o}). \tag{18.14}$$

With this, the new solution can be characterized by the potentials

$$Z = 1, \qquad H = \tanh F \tag{18.15}$$

and the metric and field components can conveniently be expressed in the form

$$\begin{aligned} &U = U_\mathrm{o}, && e^{-V} = e^{-U}\cosh^2 F, \\ &W = 0, && e^{-M} = e^{2F}\cosh^2 F e^{-M_\mathrm{o}}, \\ &\Phi_0^\circ = -\mathrm{sech}\, F F_v, && \Phi_2^\circ = \mathrm{sech}\, F F_u. \end{aligned} \tag{18.16}$$

Chandrasekhar and Xanthopoulos (1987*a*) have also obtained the interesting result that if, using the transformation (18.16), the initial solution is chosen such that

$$F = \tfrac{1}{2}\sum_{n=0}^{\infty} a_n P_n(t)P_n(z) - Q_0(t)P_0(z) \tag{18.17}$$

then, in the new solution, the surface on which $t=1$ does not correspond to a curvature singularity as in the Bell–Szekeres solution and the solution of Section 17.3.

Using the same definition (18.14), it is also possible to show that the more general solution obtained from (17.32) can be written in the form

$$\begin{aligned} &U = U_\mathrm{o}, && e^{-V} = \frac{1}{(a^2+c^2)} e^{-U_\mathrm{o}}\big((1-b)\cosh F - a\sinh F\big)^2, \\ &W = 0, && e^{-M} = e^{2F}\big((1-b)\cosh F - a\sinh F\big)^2 e^{-M_\mathrm{o}}, \\ &\Phi_0^\circ = -\frac{(c+ad-bc)F_v}{(1-b)\cosh F - a\sinh F}, && \Phi_2^\circ = \frac{(c+ad-bc)F_v}{(1-b)\cosh F - a\sinh F}. \end{aligned} \tag{18.18}$$

It may be noted that a very general solution may be obtained by taking the initial solution in the form (18.11), and using this generation technique given by (18.18) with F given by (18.14). It would appear that such general solutions would normally contain a scalar curvature singularity on the surface $t=1$. However, although this class of solutions seems to be very general, it does not necessarily contain the complete class of all possible diagonal solutions, even for the restricted case in which Φ_0, Φ_2 and thus H or η are all real.

18.3 Electromagnetic Gowdy cosmologies

It has already been pointed out in Section 10.7 that the Gowdy cosmologies, which contain interacting plane waves, can under certain conditions also be made to satisfy the necessary boundary conditions appropriate for their interpretation in terms of a collision of initial plane waves. The extension to the electromagnetic Gowdy cosmologies has been carried out by Charach (1979). A reinterpretation of these solutions as the interaction following the collision of plane electromagnetic waves is now presented in the above notation.

It is first appropriate to re-introduce the alternative coordinate system (10.58) defined by

$$\tilde t = f + g, \qquad \tilde z = f - g \tag{18.19}$$

where $\tilde t$ is a past pointing, time-like coordinate which decreases towards the singularity when $\tilde t = 0$ in region IV. With this, the line element for the diagonal metric is given by (10.59), and the main field equations (18.6) take the form

$$\ddot V + \frac{1}{\tilde t}\dot V - V'' + \frac{2}{\tilde t}e^{-V}(\dot H\dot{\bar H} - H'\bar H') = 0 \tag{18.20a}$$

$$\dot H\bar H' - H'\dot{\bar H} = 0 \tag{18.20b}$$

$$\ddot H - H'' - \dot V\dot H + V'H' = 0 \tag{18.20c}$$

where the dot and prime denote derivatives with respect to $\tilde t$ and $\tilde z$ respectively.

To express these equations in the form of Carach (1979), it is convenient first to put

$$V = 2\psi - \log\tilde t, \qquad \text{or} \qquad \psi = \tfrac{1}{2}(V - U). \tag{18.21}$$

It is then possible to put

$$\dot H = -\dot\rho + \frac{i}{\tilde t}e^{2\psi}\sigma', \qquad H' = -\rho' + \frac{i}{\tilde t}e^{2\psi}\dot\sigma \tag{18.22}$$

where ρ and σ are the two non-zero components of the four-potential for the electromagnetic field. With them the non-zero components of the electromagnetic field tensor are given by

$$\begin{aligned} F_{14} &= -F_{41} = \dot{\sigma}, \qquad & F_{24} &= -F_{42} = \sigma', \\ F_{13} &= -F_{31} = \dot{\rho}, \qquad & F_{23} &= -F_{32} = \rho'. \end{aligned} \tag{18.23}$$

With these substitutions, the main field equations together with Maxwell's equations take the form

$$\begin{aligned} \ddot{\psi} + \frac{1}{\tilde{t}}\dot{\psi} - \psi'' &= -e^{-2\psi}(\dot{\rho}^2 - {\rho'}^2) + \frac{1}{\tilde{t}^2}e^{2\psi}(\dot{\sigma}^2 - {\sigma'}^2) \\ \dot{\rho}\dot{\sigma} - \rho'\sigma' &= 0 \\ \ddot{\rho} + \frac{1}{\tilde{t}}\dot{\rho} - \rho'' &= 2(\dot{\psi}\dot{\rho} - \psi'\rho') \\ \ddot{\sigma} - \frac{1}{\tilde{t}}\dot{\sigma} - \sigma'' &= 2(\dot{\psi}\dot{\sigma} - \psi'\sigma'). \end{aligned} \tag{18.24}$$

It may be noticed that the imaginary part of (18.20c) is automatically satisfied, and that the fourth equation in (18.24) enters as the integrability condition for the definitions (18.22).

Charach has presented a family of exact solutions of these equations for which ψ and ρ are functionally related by the equation

$$\psi = \tfrac{1}{2}\log P(\rho), \qquad \text{and} \qquad \sigma = 0. \tag{18.25}$$

In this case, $H = -\rho$, and the equations (18.24) become

$$\begin{aligned} \ddot{\rho} + \frac{1}{\tilde{t}}\dot{\rho} - \rho'' - \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}\rho}(\dot{\rho}^2 - {\rho'}^2) &= 0 \\ \left(\frac{\mathrm{d}^2P}{\mathrm{d}\rho^2} + 2\right)(\dot{\rho}^2 - {\rho'}^2) &= 0. \end{aligned} \tag{18.26}$$

It may be assumed that $\dot{\rho} \neq \pm\rho'$ as this does not lead to solutions with interacting waves. It follows then that P can be written in the form

$$P = c_1 + c_2\rho - \rho^2 \tag{18.27}$$

where c_1 and c_2 are arbitrary constants. The non-linearity in the resulting equation can then be removed by the substitution

$$\rho = \rho_{\mathrm{o}} - a\tanh aF \tag{18.28}$$

provided

$$c_1 = a^2 - \rho_o^2, \qquad c_2 = 2\rho_o \tag{18.29}$$

and F is a solution of the linear equation

$$\ddot{F} + \frac{1}{\tilde{t}}\dot{F} - F'' = 0. \tag{18.30}$$

It may immediately be noticed that (18.30) is identical to the vacuum equation (10.60) as may also be deduced from (18.20a). This equation may also be written in terms of different coordinates in the form of the vacuum equations (10.2) or (9.3) as

$$2F_{uv} - U_u F_v - U_v F_u = 0. \tag{18.31}$$

The resulting metric function is then given by $e^{-V} = e^{-U} a^{-2} \cosh^2 aF$.

The method described above can clearly be used to generate electromagnetic solutions from any initial vacuum solution. It may be noticed, however, that the constant ρ_o can be removed from the electromagnetic potential, and that it is possible to set $a = 1$ without loss of generality. In this case the solutions are characterized by the potentials

$$Z = 1, \qquad H = \tanh F \tag{18.32}$$

which can be seen to be identical to (18.15). This solution-generating technique is thus identical to that of Chandrasekhar and Xanthopoulos (1987a) as described at the end of the previous section. In terms of the function F, the metric and field components are given by (18.16).

Charach (1979) has presented the class of solutions in which

$$\begin{aligned} F = & - a \log \tilde{t} + \sum_\omega \{ A_\omega \cos[\omega(\tilde{z} + \alpha_\omega)] J_0(\omega \tilde{t}) \} \\ & + \sum_\omega \{ B_\omega \cos[\omega(\tilde{z} + \beta_\omega)] Y_0(\omega \tilde{t}) \} \end{aligned} \tag{18.33}$$

where $J_0(n\tilde{t})$ and $Y_0(n\tilde{t})$ are Bessel functions of the first and second kinds of zero order, and A_n, B_n, α_n and β_n are sets of arbitrary constants. This has been obtained by separating the variables in (18.30), and considering only solutions that are periodic in $\tilde{z}$.

As pointed out by Feinstein and Ibañez (1989) and described here in Section 10.7, these solutions can easily be adapted to the boundary conditions for colliding plane waves if F also includes the terms

$$F = \sum_n d_n \cosh^{-1}\left(\frac{\tilde{z} + c_n}{\tilde{t}} \right). \tag{18.34}$$

It may also be noted that, since the equation (18.30) or (18.31) for F is the by now familiar Euler–Poisson–Darboux equation, further explicit solutions can be obtained using any combination of the terms described previously in Section 10.8.

18.4 Other methods

Further diagonal solutions which describe colliding plane electromagnetic waves can easily be generated from known solutions, and also from known colliding plane gravitational wave solutions, using the invariance transformations (17.35*a*,*d*,*e*) with real constants. These transformations are generalizations of those such as (18.7) that have been considered above. Other methods are also available.

Within the context of stationary axisymmetric solutions, it is well known that a Bonnor transformation (Bonnor, 1961) can be used to relate a class of diagonal electromagnetic solutions to a class of non-diagonal vacuum solutions. The precise result may be stated in the form that if $Z_{\rm o}$ is a complex solution of the vacuum Ernst equation (11.8), then a solution of the Ernst equations (16.22) for an electromagnetic field is given by

$$Z = Z_{\rm o}\bar{Z}_{\rm o}, \qquad H = \tfrac{1}{2}(\bar{Z}_{\rm o} - Z_{\rm o}) \tag{18.35}$$

where it may be noted that here H is purely imaginary. The inverse transformation also holds.

It was first suggested by Halilsoy (1983) that this transformation could be used to generate colliding plane wave solutions. However, there is considerable difficulty with the application of the boundary conditions. If $Z_{\rm o}$ satisfies the junction conditions (11.26) for colliding non-aligned gravitational plane waves, then (Z, H) given by (18.35) do not necessarily satisfy the junction conditions (16.28) for colliding aligned electromagnetic waves. All is not lost, however, since it is possible to start with solutions of (11.8) or (16.22) which can not be interpreted as colliding plane waves, and then to use the transformation (18.35) or its inverse to generate solutions which may be so interpreted for certain values of the free parameters.

In view of the large number of solutions of Ernst's equation that are now known, such an approach to the generation of colliding plane wave solutions is feasible. However, it has not yet been used effectively.